

BRANCHING LAWS FOR DISCRETE WALLACH POINTS

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ABSTRACT. We consider the (projective) representations of the group of holomorphic automorphisms of a symmetric tube domain $V \oplus i\Omega$ that are obtained by analytic continuation of the holomorphic discrete series. For a representation corresponding to a discrete point in the Wallach set, we find the decomposition under restriction to the identity component of $GL(\Omega)$. Using Riesz distributions, an explicit intertwining operator is constructed as an analytic continuation of an integral operator. The density for the Plancherel measure involves quotients of Γ -functions and the c -function for a symmetric cone of smaller rank.

1. INTRODUCTION

Let \underline{G} be the identity component of the group of biholomorphisms of a irreducible bounded symmetric domain \underline{D} . The scalar holomorphic discrete series of \underline{G} can be realised in the space of holomorphic functions on this domain. By reproducing kernel techniques, M. Vergne and H. Rossi [VR76] have shown (see also [Ber75, Wal79, FK94]) that it has an analytic continuation as a family of (projective) irreducible unitary representations π_α of \underline{G} , parametrised by the so-called Wallach set. Let r be the rank of the domain and d its characteristic number (cf. next section for a definition). Then the Wallach set is the union of the half-line $\alpha > (r-1)\frac{d}{2}$ and a discrete part consisting of r points $l\frac{d}{2}$, $l = 0, \dots, r-1$. When $\alpha > p-1$, where p is the genus of \underline{D} , the representation spaces are weighted Bergman spaces.

Let τ be an antilinear involution of \underline{D} . Then $\mathcal{D} := \underline{D}^\tau$ is a totally geodesic submanifold, hence a Riemannian symmetric space, and $G := \underline{G}^\tau$ contains its group of displacements. Such a domain is called a real bounded symmetric domain.

When one restricts an irreducible unitary representation of a group to a subgroup, the representation need not to be irreducible anymore, and the decomposition into irreducibles is called a branching law. In our context two branching problems have been extensively studied: the decomposition of the tensor product representation $\pi_\alpha \hat{\otimes} \bar{\pi}_\alpha$ and the restriction of π_α to symmetric subgroups $G = \underline{G}^\tau$ where τ is an antilinear involution of \mathcal{D} . A formula for the first problem and for $\alpha > p-1$ was given without proof by Berezin for classical domains in [Ber78]. H. Upmeyer and A. Unterberger extended it to

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all domains and gave a Jordan theoretic proof [UU94]. The second problem was solved (for the same parameters) by G. Zhang and (independently) by G. van Dijk and M. Pevzner [Zha01a, vDP01], and also by Y. Neretin for classical groups [Ner00a]. Those two problems are in fact similar. The restriction map from $\underline{\mathcal{D}}$ to \mathcal{D} (resp. from $\underline{\mathcal{D}} \times \underline{\mathcal{D}}$ to $\underline{\mathcal{D}}$) gives rise to the Berezin transform on \mathcal{D} (resp. $\underline{\mathcal{D}}$), which is a kernel operator. The solution then consists in computing the spectral symbol of the Berezin transform, or, if one prefers, in computing the Fourier transform of the Berezin kernel. In [Zha00] and in [vDP01, Section 5] the problem of decomposing $\pi_\alpha \widehat{\otimes} \overline{\pi}_{\alpha+l}$ where $l \in \mathbb{N}$ is also solved, by the same method. A similar problem is also studied in [FP05].

For arbitrary parameters, those problems are more complicated, and no general method seems to apply. In [ØZ97] the tensor product problem for $\underline{G} = SU(2, 2)$ is solved for any parameter. In [Zha01b] the representation $\pi_{\frac{d}{2}} \widehat{\otimes} \overline{\pi}_{\frac{d}{2}}$ is decomposed for any \underline{G} ($\pi_{\frac{d}{2}}$ is called the minimal representation). In [Ner02], Y. Neretin solves the restriction problem from $U(r, s)$ to $O(r, s)$ ($r \leq s$) for any parameter by analytic continuation of the result for large parameters. If $r = s$ the support of the Plancherel formula remains the same for all $\alpha > r - 1$ (here $d = 2$) but when $s - r$ is sufficiently large new pieces appear when α crosses $p - 1 = 2(r + s) - 1$ and the situation gets worse as α approaches to $(r - 1)\frac{d}{2}$, as he had already explained in [Ner00b]. For points in the discrete Wallach set, the situation is not clear. In his thesis the second author manages to decompose the restriction of $SO(2, n)$ to $SO(1, n)$ for any parameter [Sep07b], as well as the restriction of the minimal representation of $SU(p, q)$ to $SO(p, q)$ [Sep07a], and the minimal representation of $Sp(n, \mathbb{R})$ (resp. $SU(n, n)$) to $GL^+(n, \mathbb{R})$ (resp. to $GL(n, \mathbb{C})$) [Sep08].

Assume that \mathcal{D} is of tube type, i.e. that \mathcal{D} is biholomorphic to the tube domain T_Ω over the symmetric cone Ω . Then the inverse image of Ω is a real bounded symmetric domain. In this paper, generalising [Sep08], we establish, for any parameter in the discrete Wallach set, the branching rule for the restriction of the associated representation of $\underline{G} = G_0(T_\Omega)$ to $G = GL_0(\Omega)$.

We use the model by Rossi and Vergne which realises the representation given by the l -th point in the Wallach set as $L^2(\partial_l \Omega, \mu_l)$, where $\partial_l \Omega$ is the set of positive semidefinite elements in $\partial \Omega$ of rank l , and μ_l is a relatively G -invariant measure on $\partial_l \Omega$. A key observation is that for any x in $\partial_l \Omega$, the function $g \mapsto \Delta_\nu(g^*x)$ on G , where Δ_ν is the power function of the Jordan algebra, transforms like a function in a certain parabolically induced representation. A naive approach to construct an intertwining operator from $L^2(\partial_l \Omega, \mu_l)$ into a direct sum of parabolically induced representations would then to weight the functions above by compactly supported smooth functions, i.e., to consider mappings $f \mapsto \int_{\partial_l \Omega} f(x) \Delta_\nu(g^*x) d\mu_l(x)$, for f in $C_0^\infty(\partial_l \Omega)$. It will become clear that this approach is in fact fruitful. However, there are two problems that have to be dealt with. First of all, it is not obvious that the natural target spaces are unitarisable. Secondly, and more importantly, the integrals above need not converge for the suitable choice of parameters ν . However, as we shall see, both these problems can be solved.

The paper is organised as follows. In Section 2 we recall some facts about Jordan algebras and symmetric cones that will be needed in the paper. In Section 3 we prove an identity between the restriction of a spherical function for the cone Ω to a cone of lower rank in its boundary and the corresponding spherical function for the lower rank cone. In Section 4 we define a class of irreducible unitary spherical representations that provides target spaces for the integral operators discussed above. These are constructed using the Levi decomposition of the group G by twisting parabolically induced unitary representations for the semisimple factor of G by a certain character. In Section 5 we construct the intertwining operator as an analytic continuation of the integral operator above. After this has been taken care of, a polar decomposition for the measure μ_l due to J. Arazy and H. Upmeyer [AU03] allows to express the restriction of the intertwining operator to K -invariant vectors in terms of the Fourier transform for a cone of rank l . Using this identification, the inversion formula for the Fourier transform can be used to prove the Plancherel theorem for the branching problem. In the appendix we provide a framework for certain restrictions of distributions to submanifolds which will be useful for giving an analytic continuation for the integral that should give an intertwining operator. It should be pointed out that the standard theory for restricting distributions (e.g. [Hör83, Cor. 8.2.7.]) does not apply to our situation since the condition on the wave front set for the distribution is not satisfied. Instead we have to use restrictions based on extending test functions in such a way that they are constant in certain directions from the submanifold (cf. Appendix A).

We finally want to mention that branching problems related to holomorphic involutions of $\underline{\mathcal{D}}$ have also been studied in [Rep79, Kob98, BS02, PZ04].

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2. JORDAN THEORETIC PRELIMINARIES

Let V be a Euclidean Jordan algebra. It is a commutative real algebra with unit element e such that the multiplication operator $L(x)$ satisfies $[L(x), L(x^2)] = 0$, and provided with a scalar product for which $L(x)$ is symmetric. An element is invertible if its quadratic representation $P(x) = 2L(x)^2 - L(x^2)$ is so. Its cone of invertible squares Ω is a symmetric cone: it is homogeneous under the identity component, G , of the Lie group $GL(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\}$, and it is self-dual. It follows that the involution $\Theta(g) := g^{-*} := (g^*)^{-1}$ (where g^* is the adjoint of g with respect to the scalar product of V) preserves G (which is hence reductive). The stabiliser $K = G_e$ of e coincides with the identity component of the group $\text{Aut}(V)$ of automorphisms of V and with the fixed points of G under the involution Θ , and hence is compact. Thus Ω is a Riemannian symmetric space.

The tube $T_\Omega = V \oplus i\Omega$ over Ω in the complexification of V is an Hermitian symmetric space of the non-compact type, diffeomorphic via the Cayley transform to a (tube type) bounded symmetric domain. Any element of

$GL(\Omega)$, when extended complex-linearly, preserves T_Ω . In this fashion G is seen as a subgroup of the identity component \underline{G} of the group of biholomorphisms of T_Ω .

We assume that V is simple. Then there exists a positive integer r , called the rank of V , such that any family of mutually orthogonal minimal idempotents has r elements. Such a family is called a Jordan frame. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of the Lie algebra \mathfrak{g} of G . Then the map $x \mapsto L(x)$ yields an isomorphism $V \rightarrow \mathfrak{p}$. The subspace generated by a Jordan frame (more precisely by the associated multiplication operators) is a maximal abelian subspace of \mathfrak{p} and conversely, any maximal abelian subspace of \mathfrak{p} determines (up to order) a Jordan frame. From now we fix a choice of a Jordan frame (c_1, \dots, c_r) and let

$$\mathfrak{a} = \langle L(c_j), j = 1, \dots, r \rangle$$

and $A = \exp \mathfrak{a}$.

Any x in V can be written

$$x = k \sum_{1 \leq j \leq r} \lambda_j c_j$$

where $k \in K$ and the λ_j are real numbers, and the family $(\lambda_1, \dots, \lambda_r)$ is unique up to permutation (its members are called the eigenvalues of x). Then x belongs to Ω if and only if for all $1 \leq j \leq r$, $\lambda_j > 0$, and this spectral decomposition corresponds the KAK decomposition of G , the A -component in the decomposition being unique up to conjugation by an element of the Weyl group $W = \mathfrak{S}_r$ of G . The rank of x is defined to be the number of its nonzero eigenvalues. There exists on V a K -invariant polynomial function $\Delta(x)$ (the determinant) and a K -invariant linear function $\text{tr}(x)$ (the trace) that satisfy

$$\Delta(x) = \prod_{j=1}^r \lambda_j \quad \text{and} \quad \text{tr}(x) = \sum_{j=1}^r \lambda_j.$$

The determinant defines the character

$$(1) \quad \Delta(g) := \Delta(ge)$$

of the group G .

A Jordan frame gives rise to the important Peirce decomposition. Since multiplications by orthogonal idempotents commute, the space V decomposes into a direct sum of joint eigenspaces for the (symmetric) operators $(L(c_j))_{j=1, \dots, r}$. The eigenvalues of $L(c)$ when c is an idempotent, belong to $\{0, \frac{1}{2}, 1\}$. Let us denote by $V(c, \alpha)$ the eigenspace corresponding to the value α . The decomposition into joint eigenspaces is then given by

$$V = \bigoplus_{1 \leq i \leq j \leq r} V_{ij},$$

where

$$V_{ii} = V(c_i, 1) \cap \bigcap_{j \neq i} V(c_j, 0),$$

and when $i \neq j$,

$$V_{ij} = V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}) \cap \bigcap_{k \notin \{i,j\}} V(c_k, 0).$$

We have $V_{ii} = \mathbb{R}c_i$ and the V_{ij} all have the same dimension d , called the degree of the Jordan algebra.

We can now describe the roots of $(\mathfrak{g}, \mathfrak{a})$. Let $(\delta_j)_{j=1,\dots,r}$ be the dual basis of $(L(c_j))_{j=1,\dots,r}$ in \mathfrak{a}^* . Then the roots are

$$\alpha_{ij}^\pm = \pm \frac{\delta_j - \delta_i}{2}, \quad 1 \leq i < j \leq r,$$

and the corresponding root spaces are

$$\begin{aligned} \mathfrak{g}_{ij}^+ &= \{a \square e_i \mid a \in V_{ij}\}, \\ \mathfrak{g}_{ij}^- &= \{a \square e_j \mid a \in V_{ij}\}. \end{aligned}$$

where $x \square y = L(L(x)y) + [L(x), L(y)]$. Let N be the nilpotent subgroup

$$N = \exp \bigoplus_{1 \leq i < j \leq r} \mathfrak{g}_{ij}^+.$$

Then G has the Iwasawa decomposition $G = NAK$.

For any idempotent c , the projection on $V(1, c)$ is $P(c)$, and $V(1, c)$ is a Jordan subalgebra, hence a Euclidean Jordan algebra with neutral element c (note that it is simple with rank the one of c). We denote by $\Omega_1(c)$ its symmetric cone. In particular for

$$e_j = \sum_{k=1}^l c_k.$$

we set

$$V^{(l)} = V(1, e_l) \quad \text{and} \quad \Omega^{(l)} = \Omega_1(e_l),$$

and also note $G^{(l)}$ the identity component of $G(\Omega^{(l)})$, $K^{(l)} = G_{e_l}$ and $\Delta^{(l)}$ the determinant of $V^{(l)}$. The principal minors of V are then defined by the formula

$$\Delta_{(j)}(x) := \Delta^{(j)}(P(e_j)(x)).$$

Then x is in Ω if and only if for all $1 \leq j \leq r$, $\Delta_{(j)}(x) > 0$. Let $\nu \in \mathbb{C}^r$ and set for x in Ω ,

$$\Delta_\nu(x) = \Delta_{(1)}^{\nu_1 - \nu_2}(x) \Delta_{(2)}^{\nu_2 - \nu_3}(x) \dots \Delta_{(r-1)}^{\nu_{r-1} - \nu_r}(x) \Delta_{(r)}^{\nu_r}(x).$$

Using the basis (δ_j) , we can identify ν with an element of $\mathfrak{a}_{\mathbb{C}}^*$. Then if $a(g)$ is the projection of g on A in the Iwasawa decomposition,

$$(2) \quad \Delta_\nu(gx) = e^{\nu \log a(g)} \Delta_\nu(x).$$

The action of G on the boundary $\partial\Omega$ of Ω has $r - 1$ orbits, which may be parametrised by the rank of its elements. We denote by $\partial_l\Omega$ the orbit of rank l elements, i.e.,

$$\partial_l\Omega = Ge_l.$$

There exists on $\partial_l\Omega$ a unique relatively G -invariant measure μ_l , which transforms according to

$$d\mu_l(gx) = \Delta^{\frac{ld}{2}}(g) d\mu_l(x).$$

The Hilbert space associated to the Wallach point $l\frac{d}{2}$ is, up to renormalisation, isometric to $L^2(\partial_l\Omega, \mu_l)$ [FK94, Theorem X.III.4], and the representation of G in this picture is then given by

$$(3) \quad \pi^l(g)f = \Delta^{\frac{ld}{4}}(g)f(g^*\cdot).$$

The measures μ_l were constructed by M. Lassalle [Las87] and can also be obtained as Riesz distribution, thanks to S. Gindikin's theorem [FK94, VII.3]. A major tool for our purpose will be the polar decomposition of μ_l [AU03, Theorem 3.2.6]. Let $\Pi_l = K.e_l$ be the set of idempotents of rank l . Then $\partial_l\Omega$ is the disjoint union

$$\partial_l\Omega = \bigsqcup_{u \in \Pi_l} \Omega(u).$$

Since elements of G permute the faces of $\overline{\Omega}$ (which are of the form $\overline{\Omega(u)}$ for idempotents u), an action is induced on Π_l , such that the preceding equality defines a G -equivariant fibration

$$\partial_l\Omega \rightarrow \Pi_l.$$

For any function f in the space $C_0^\infty(\partial_l\Omega)$ of smooth functions with compact support on $\partial_l\Omega$,

$$(4) \quad \int_{\partial_l\Omega} f d\mu_l = \int_K dk \int_{\Omega^{(l)}} \Delta^{\frac{rd}{2}}_{(l)}(x) f(kx) d_*^{(l)}x,$$

where $d_*^{(l)}x$ is the unique $G^{(l)}$ -invariant measure on $\Omega^{(l)}$.

The set $\partial_l\Omega$ is not a submanifold of V . However let $V_{\geq l}$ be the (open) set of elements in V with rank bigger or equal than l . To any l -element subset $I_l \subset \{1, \dots, r\}$ one can associate the idempotent $e_{I_l} = \sum_{j \in I_l} c_j$ and the minor $\Delta_{I_l}(x) := \Delta(P(e_{I_l})x + e - e_{I_l})$. Then

$$V_{\geq l} = \bigcup_{I_l \subset \{1, \dots, r\}} \{x \in V \mid \Delta_{I_l}(x) \neq 0\},$$

and [Las87, Propositions 3 and 7] show that $\partial_l\Omega$ is a (closed) submanifold of $V_{\geq l}$.

3. AN IDENTITY BETWEEN SPHERICAL FUNCTIONS

The spherical functions on Ω may be defined for ν in $\mathfrak{a}_{\mathbb{C}}^*$ by the formula

$$\Phi_\nu(x) = \int_K \Delta_\nu(kx) dk.$$

When ν satisfies $\Re \nu_1 \geq \dots \geq \Re \nu_r \geq 0$, a property that we will denote by $\Re \nu \geq 0$, the generalised power function Δ_ν and the spherical function Φ_ν extend continuously to $\overline{\Omega}$. Now let

$$(5) \quad \mathfrak{a}_l = \langle L(c_j), j = 1, \dots, l \rangle$$

for $1 \leq l \leq r-1$ and assume that ν belongs to $(\mathfrak{a}_{\mathbb{C}})^*$, i.e. that its $(r-l)$ last coordinates vanish. Then ν also defines a spherical function $\Phi_\nu^{(l)}$ of $\Omega^{(l)}$. Let α be a real number. When ν appears in the argument of an object related to $V^{(l)}$, we will use the convention that $\nu + \alpha := (\nu_1 + \alpha, \dots, \nu_l + \alpha)$. Recall that $\Omega^{(l)} \subset \partial_l\Omega$.

Theorem 3.1. *Let ν in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\Re \nu \geq 0$. Then for all x in $\Omega^{(l)}$,*

$$\Phi_{\nu}(x) = \gamma_{\nu}^{(l)} \Phi_{\nu}^{(l)}(x), \quad \text{where} \quad \gamma_{\nu}^{(l)} = \frac{\Gamma_{\Omega^{(l)}}(\frac{rd}{2}) \Gamma_{\Omega^{(l)}}(\nu + \frac{ld}{2})}{\Gamma_{\Omega^{(l)}}(\frac{ld}{2}) \Gamma_{\Omega^{(l)}}(\nu + \frac{rd}{2})}.$$

Here $\Gamma_{\Omega^{(l)}}$ is the Gindikin Gamma function for the cone $\Omega^{(l)}$,

$$\Gamma_{\Omega^{(l)}}(\nu) = (2\pi)^{\frac{l(l-1)d}{4}} \prod_{j=1}^l \Gamma(\nu_j - (j-1)\frac{d}{2}).$$

The theorem is proved in the case $\nu \in \mathbb{N}^l$ in [AU03, Proposition 1.3.2 and remark 1.3.4]. We use this result and the following lemma, which is based on Blaschke's theorem (see [Kr01, Lemma A.1] for a detailed proof).

Lemma 3.2. *Let f be a holomorphic function defined on the right half-plane $\{z \in \mathbb{C} \mid \Re z > 0\}$. If f is bounded and $f(n) = 0$ for $n \in \mathbb{N}$, then f is identically zero.*

Proof of the theorem. Let us set $z_j = \nu_j - \nu_{j+1}$, $j = 1, \dots, l-1$, $z_l = \nu_l$, so that $\Re z_j \geq 0$ and $\nu_j = \sum_{k=j}^l z_k$. Let $x \in \Omega^{(l)}$ and let

$$F(z_1, \dots, z_l) = \Phi_{\nu(z_1, \dots, z_l)}(x) - \gamma_{\nu(z_1, \dots, z_l)}^{(l)} \Phi_{\nu}^{(l)}(x).$$

Let us fix $z_j = m_j \in \mathbb{N}$, $j = 2, \dots, l$. If $b \geq a > 0$, one can see by Stirling's formula that $\frac{\Gamma(z+a)}{\Gamma(z+b)}$ is bounded on the right half plane. It follows that the function

$$z \mapsto \frac{\Gamma(z + \sum_{k=2}^l m_k + \frac{ld}{2})}{\Gamma(z + \sum_{k=2}^l m_k + \frac{rd}{2})}$$

is bounded and hence also $z \mapsto \gamma_{\nu(z, m_2, \dots, m_l)}^{(l)}$. Now

$$\begin{aligned} |\Phi_{\nu(z, m_2, \dots, m_l)}(x)| &\leq \int_K \left| \Delta_{(1)}^z(kx) \Delta_{(2)}^{m_2}(kx) \dots \Delta_{(l)}^{m_l}(kx) \right| dk \\ &\leq \sup_K \left(\Delta_{(2)}^{m_2}(kx) \dots \Delta_{(l)}^{m_l}(kx) \right) \left(\sup_K \Delta_{(1)}(kx) \right)^{\Re(z)}, \end{aligned}$$

and

$$\left| \Phi_{\nu(z, m_2, \dots, m_l)}^{(l)}(x) \right| \leq \sup_{K^{(l)}} \left(\Delta_{(2)}^{m_2}(kx) \dots \Delta_{(l)}^{m_l}(kx) \right) \left(\sup_{K^{(l)}} \Delta_{(1)}(kx) \right)^{\Re(z)}$$

Let $\delta > \sup_K \Delta_{(1)}(kx) \geq \sup_{K^{(l)}} \Delta_{(1)}(kx)$. Then the holomorphic function $f(z) = F(z, m_2, \dots, m_l) \delta^{-z}$ is bounded and vanishes on \mathbb{N} , hence on the right half plane, i.e., for every $z \in \mathbb{C}$ with $\Re z > 0$ and $m_j \in \mathbb{N}$,

$$F(z, m_2, \dots, m_l) = 0.$$

By the same argument one shows that for every $z_1 \in \mathbb{C}$ with $\Re z_1 > 0$ and $m_j \in \mathbb{N}$, the map $z \mapsto F(z_1, z, m_3, \dots, m_l)$ vanishes identically, and the proof follows by induction. \square

4. A SERIES OF SPHERICAL UNITARY REPRESENTATIONS

In this section we introduce a family of spherical unitary representations that will occur in the decomposition of $L^2(\partial_l \Omega)$ under the action of G .

For $1 \leq l \leq r-1$ let

$$\bar{\mathfrak{n}}_l = \bigoplus_{l \geq i < j} \mathfrak{g}_{ij}^-.$$

Note that it is a (nilpotent) Lie algebra and that (cf. (5))

$$\mathfrak{a}_l \oplus \mathbb{R}L(e) = \bigcap_{l \leq i < j} \ker \alpha_{ij}^\pm.$$

The closed subgroup $Z_G(\mathfrak{a}_l) = Z_G(\mathfrak{a}_l \oplus \mathbb{R}L(e))$ normalises $\bar{N}_l = \exp \bar{\mathfrak{n}}_l$ hence

$$Q_l = Z_G(\mathfrak{a}_l) \bar{N}_l$$

is a subgroup of G . Moreover $Z_G(\mathfrak{a}_l) \cap \bar{N}_l = \{\text{id}\}$ so this decomposition is a semidirect product. The Lie algebra of Q_l ,

$$\mathfrak{q}_l = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_l) \oplus \bar{\mathfrak{n}}_l = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{l < i < j} \mathfrak{g}_{ij}^\pm \oplus \bigoplus_{l \geq i < j} \mathfrak{g}_{ij}^-,$$

where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{t}}(\mathfrak{a})$, is a parabolic subalgebra of \mathfrak{g} , and since Q_l is the normaliser of \mathfrak{q}_l in G [Kna02, 7.83], it is a closed subgroup of G (the parabolic subgroup associated to \mathfrak{q}_l). It is also the stabiliser of a flag of idempotents (e_1, e_2, \dots, e_l) . Note also that since $V(L(c_j), 1) = \mathbb{R}c_j$, we have

$$(6) \quad Z_G(\mathfrak{a}_l)c_j = \mathbb{R}_+c_j, \quad j = 1, \dots, l.$$

Lemma 4.1. *Let $A_l = \exp \mathfrak{a}_l$ and*

$$M_l = \bigcap_{j=1}^l Z_G(\mathfrak{a}_l)_{c_j}.$$

Then the multiplication map

$$M_l \times A_l \times \bar{N}_l \rightarrow Q_l$$

is a diffeomorphism.

Proof. It is clear from (6) that the product map $M_l \times A_l \rightarrow Z_G(\mathfrak{a}_l)$ is a smooth bijection hence it is a diffeomorphism. \square

Note that the decomposition in the preceding lemma is not exactly the Langlands decomposition of Q_l . However, it is more adapted to our purpose. We will let $a_l(q)$ denote the A_l -component of $q \in Q_l$ in the preceding decomposition. For $\nu \in (\mathfrak{a}_{\mathbb{C}})^*$, let $1 \otimes e^\nu \otimes 1$ be the character of Q_l defined by $(1 \otimes e^\nu \otimes 1)(q) = e^{\nu \log a_l(q)}$, and let us denote by $C(G, Q_l, 1 \otimes e^\nu \otimes 1)$ the Frechet space of continuous complex valued functions on G that are Q_l -equivariant with respect to $1 \otimes e^\nu \otimes 1$, i.e.,

$$C(G, Q_l, 1 \otimes e^\nu \otimes 1) = \{f \in C(G) \mid \forall q \in Q_l, f(gq) = e^{-\nu \log a_l(q)} f(g)\}.$$

The induced representation $\text{Ind}_{Q_l}^G(1 \otimes e^\nu \otimes 1)$ is the left regular representation of G on $C(G, Q_l, 1 \otimes e^\nu \otimes 1)$.

We will now determine values of $\nu \in (\mathfrak{a}_{\mathbb{C}})^*$ for which the representation

$$\text{Ind}_{Q_l}^G(1 \otimes e^\nu \otimes 1) \otimes \Delta^{-\frac{ld}{4}}$$

(cf. (1)) can be made unitary and irreducible.

The group G admits the Levi decomposition

$$G = G' \times \mathbb{R}_+$$

where the semisimple part G' is the kernel of the character Δ . Then K is a maximal compact subgroup of G' and the Lie algebra \mathfrak{g}' of G' has Cartan decomposition $\mathfrak{g}' = \mathfrak{k} \oplus \mathfrak{p}'$ with $\mathfrak{p}' = \{L(x) \in \mathfrak{p} \mid \text{tr } x = 0\}$, and $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{p}'$ is maximal abelian in \mathfrak{p}' . Let

$$\begin{aligned} \mathfrak{a}'_l &= \bigoplus_{1 \leq j \leq l} \mathbb{R} \left(L(c_j) - \frac{L(e - e_l)}{r - l} \right) = \bigcap_{l \leq i < j} \ker \alpha_{ij}^\pm, \\ \mathfrak{m}'_l &= \mathfrak{m} \oplus \bigoplus_{j=l+1}^{r-1} \mathbb{R} \left(L(c_j) - \frac{L(e - e_l)}{r - l} \right) \oplus \bigoplus_{l < i < j} \mathfrak{g}_{ij}^\pm. \end{aligned}$$

Then

$$\mathfrak{q}'_l = \mathfrak{m}'_l \oplus \mathfrak{a}'_l \oplus \overline{\mathfrak{n}}_l.$$

is a parabolic subalgebra of G' . The corresponding parabolic subgroup Q'_l admits the Langlands decomposition

$$Q'_l = M'_l A'_l \overline{N}_l,$$

where $A'_l = \exp \mathfrak{a}'_l$ and (cf. [Kna02, Ch. VII, Propositions 7.25, 7.27 and 7.82])

$$M'_l = Z_K(\mathfrak{a}'_l) \exp(\mathfrak{m}'_l \cap \mathfrak{p}'),$$

whose Lie algebra is \mathfrak{m}'_l . For $\nu \in (\mathfrak{a}'_{l\mathbb{C}})^*$ the induced representation

$$\text{Ind}_{Q'_l}^{G'}(1 \otimes e^\nu \otimes 1)$$

is defined in the same way as for G .

Lemma 4.2. (i) $Q_l = Q'_l \times \mathbb{R}_+$,
(ii) $M'_l \subset M_l$.

Proof. To prove (i) we observe that since $\mathbb{R}_+ \subset Q_l$, we can write $Q_l = Q' \times \mathbb{R}_+$, for some subgroup $Q' \subset G'$. Since Q_l (resp. Q'_l) is the normaliser of \mathfrak{q}_l (resp. \mathfrak{q}'_l) in G (resp. G'), the inclusion $Q'_l \subset Q'$ is obvious and the converse follows from the fact that $\text{Ad}(G)$ preserves \mathfrak{g}' . Since $X.c_j = 0$ for $X \in \mathfrak{m}'_l$ and $1 \leq j \leq l$, the assertion (ii) will follow from $Z_K(\mathfrak{a}_l) = Z_K(\mathfrak{a}'_l)$. Let $k \in Z_K(\mathfrak{a}'_l)$, i.e., for $j = 1, \dots, l$,

$$(7) \quad k \cdot \left(L(c_j) - \frac{L(e - e_l)}{r - l} \right) = L(kc_j) - \frac{L(e - ke_l)}{r - l} = L(c_j) - \frac{L(e - e_l)}{r - l}.$$

By summing over j one obtains

$$L(ke_l) - \frac{l}{r - l} L(e - ke_l) = L(e_l) - \frac{l}{r - l} L(e - e_l),$$

and hence $L(ke_l) = L(e_l)$. By (7) we then have $L(kc_j) = L(c_j)$ for $j = 1, \dots, l$, i.e., $k \in Z_K(\mathfrak{a}_l)$. \square

Let us denote by $(\tilde{\delta}_j)_{j=1 \dots l}$ the dual basis of $(L(c_j) - \frac{L(e - e_l)}{r - l})_{j=1 \dots l}$ in $(\mathfrak{a}'_{l\mathbb{C}})^*$. By $\delta_j \mapsto \tilde{\delta}_j$ we define an isomorphism $(\mathfrak{a}_{l\mathbb{C}})^* \simeq (\mathfrak{a}'_{l\mathbb{C}})^*$, $\nu \mapsto \tilde{\nu}$.

Let m_ν be the character of \mathbb{R}_+ defined by

$$(8) \quad m_\nu(\zeta) = \zeta^{-\frac{rld}{4} + \sum_{1 \leq j} \nu_j}.$$

Proposition 4.3.

$$\mathrm{Ind}_{Q_l}^G(1 \otimes e^\nu \otimes 1) \otimes \Delta^{-\frac{dl}{4}} \simeq \mathrm{Ind}_{Q'_l}^{G'}(1 \otimes e^{\tilde{\nu}} \otimes 1) \otimes m_\nu$$

Proof. Let us denote by $a_l(q')$ the A'_l -component of q' in the Langlands decomposition of Q'_l . Then for all $q = q'\xi \in Q_l = Q'_l \times \mathbb{R}_+$,

$$(9) \quad e^{\nu \log a_l(q)} = \xi^{\sum \nu_j} e^{\tilde{\nu} \log a'_l(q')}.$$

Indeed, if $q' = m'e^{\sum s_j(L(c_j) - \frac{L(e-e_l)}{r-l})}\bar{n}$ is the Langlands decomposition of q' in Q'_l , then

$$q = (m'e^{(-\frac{l}{r-l} + \log \xi)L(e-e_l)})e^{\sum s_j L(c_j) + \log \xi L(e_l)}\bar{n}$$

is the Langlands decomposition of q in Q_l . If $f \in C(G, Q_l, 1 \otimes e^\nu \otimes 1)$, then by (9), its restriction \tilde{f} to G' belongs to the space $C(G', Q'_l, 1 \otimes e^{\tilde{\nu}} \otimes 1)$. Conversely, if $\tilde{f} \in C(G', Q'_l, 1 \otimes e^{\tilde{\nu}} \otimes 1)$, one obtains, again by (9), a function $f \in C(G, Q_l, 1 \otimes e^\nu \otimes 1)$ by setting $f(g'\zeta) := \zeta^{-\sum \nu_j} \tilde{f}(g')$ and we obtain thereby a bijection $C(G, Q_l, 1 \otimes e^\nu \otimes 1) \simeq C(G', Q'_l, 1 \otimes e^{\tilde{\nu}} \otimes 1)$. Now the operator

$$\begin{aligned} \mathcal{T} : C(G, Q_l, 1 \otimes e^\nu \otimes 1) \otimes \mathbb{C} &\rightarrow C(G', Q'_l, 1 \otimes e^{\tilde{\nu}} \otimes 1) \otimes \mathbb{C}, \\ f \otimes z &\mapsto \tilde{f} \otimes z, \end{aligned}$$

intertwines the actions of G . Indeed, if $f \in C(G, Q_l, 1 \otimes e^\nu \otimes 1)$ and $g = g'\zeta \in G = G' \times \mathbb{R}_+$ then $\widetilde{f(g^{-1}\cdot)} = \widetilde{f(g'^{-1}\zeta^{-1}\cdot)} = \zeta^{\sum \nu_j} \tilde{f}(g'^{-1}\cdot)$, and hence

$$\begin{aligned} \mathcal{T} \left(f(g^{-1}\cdot) \otimes \Delta^{-\frac{dl}{4}}(g)z \right) &= \widetilde{f(g^{-1}\cdot)} \otimes \Delta^{-\frac{ld}{4}}(g)z \\ &= \zeta^{\sum \nu_j} \tilde{f}(g'^{-1}\cdot) \otimes \zeta^{-\frac{rld}{4}}z \\ &= \tilde{f}(g'^{-1}\cdot) \otimes \zeta^{-\frac{rld}{4} + \sum \nu_j}z. \end{aligned}$$

□

The representation $\mathrm{Ind}_{Q_l}^G(1 \otimes e^\nu \otimes 1) \otimes \Delta^{-\frac{ld}{4}}$ extends to a continuous representation (denoted by the same symbol) on the Hilbert completion of $C(G, Q_l, 1 \otimes e^\nu \otimes 1) \otimes \mathbb{C}$ with respect to

$$|f \otimes z|^2 = \int_K f(k)dk |z|^2.$$

Lemma 4.4. *For any $m \in M_l$ and any $x \in V$,*

$$\Delta_{(j)}(mx) = \Delta_{(j)}(x), \quad 1 \leq j \leq l.$$

Proof. Let $m \in M_l$. Then for $j = 1, \dots, l$, m commutes with $L(e_j)$, and $m \in \mathrm{Aut} V^{(j)}$, so

$$\begin{aligned} \Delta_{(j)}(mx) &= \Delta^{(j)}(P(c_j)(mx)) = \Delta^{(j)}(mP(c_j)(x)) = \Delta^{(j)}(P(c_j)(x)) \\ &= \Delta_{(j)}(x). \end{aligned}$$

□

Proposition 4.5. *The map $g \mapsto \Delta_{-\nu}(g^*e)$ is a (norm one) K -invariant vector in $C(G, Q_l, 1 \otimes e^\nu \otimes 1)$.*

Proof. Let $q \in Q_l$ and let $q = man \in M_l A_l \overline{N}_l$. Then for $g \in G$, one has, since $n^* \in N$ and $a^* = a$,

$$\Delta_{-\nu}((gq)^*e) = \Delta_{-\nu}(n^*a^*m^*g^*e) = e^{-\nu \log(a)} \Delta_{-\nu}(m^*g^*e)$$

and because of Lemma 4.4,

$$\Delta_{-\nu}((gq)^*e) = e^{-\nu \log(a)} \Delta_{-\nu}(g^*e).$$

□

Let

$$\rho_l = \frac{d}{2} \sum_{l \geq i < j} \frac{\delta_i - \delta_j}{2} = \frac{d}{4} \sum_{j=1}^l (r+1-2j)\delta_j$$

be the half sum of the negative $\mathfrak{a}_l \oplus \mathbb{R}L(e)$ -restricted roots (counted with multiplicities).

Theorem 4.6. *For almost every $\lambda \in \mathfrak{a}_l^*$ (with respect to Lebesgue measure), the representation $\text{Ind}_{Q_l}^G(1 \otimes e^\nu \otimes 1) \otimes \Delta^{-\frac{ld}{4}}$ with $\nu = i\lambda + \rho_l + \frac{ld}{4}$ is an irreducible unitary spherical representation.*

Proof. First, let us remark that if $\nu = i\lambda + \rho_l + \frac{ld}{4}$ with $\lambda \in \mathfrak{a}_l^*$, then since

$$\sum \nu_j - \frac{rld}{4} = i \sum \lambda_j + \frac{d}{4} \left(l(r+1+l) - 2 \frac{l(l+1)}{2} - rl \right) = i \sum \lambda_j,$$

the representation m_ν (cf. (8)) is unitary. We now claim that $\widetilde{\rho_l + \frac{ld}{4}}$ is the half sum of the negative \mathfrak{a}_l' -restricted roots (counted with multiplicities). Indeed,

$$\begin{aligned} & \frac{d}{2} \sum_{l \geq i < j} \frac{\delta_i - \delta_j}{2} \left(L(e_k) - \frac{L(e - e_l)}{r-l} \right) \\ &= \frac{d}{4} (r+1-2k) + \frac{d}{2} \sum_{i \leq l < j} \frac{\delta_i - \delta_j}{2} \left(-\frac{L(e - e_l)}{r-l} \right) \\ &= \frac{d}{4} (r+1-2k) + \frac{d}{4} \sum_{i \leq l < j} (-\delta_j) \left(-\frac{1}{r-l} \sum_{k>l} L(c_k) \right) \\ &= \frac{d}{4} (r+1-2k) + \frac{ld}{4} = \widetilde{\rho_l + \frac{ld}{4}} \left(L(e_k) - \frac{L(e - e_l)}{r-l} \right). \end{aligned}$$

The theorem now follows from Proposition 4.3 and Bruhat's theorem [vdB97, Theorem 2.6]. □

Let us note $\rho'_l = \rho_l + \frac{ld}{4}$. We now set

$$(\pi_\nu, \mathcal{H}_\nu) := \text{Ind}_{Q_l}^G(1 \otimes e^{i\nu + \rho'_l} \otimes 1) \otimes \Delta^{-\frac{ld}{4}},$$

and

$$v_\nu := \Delta_{-(i\nu + \rho'_l)}((\cdot)^*e) \otimes 1.$$

Let us compute the positive definite spherical function associated to π_ν , that is,

$$\Phi(g) = \langle \pi_\nu(g)v_\nu, v_\nu \rangle_\nu = \int_K \Delta_{-(i\nu + \rho'_l)}((g^{-1}k)^*e) dk \Delta^{-\frac{ld}{4}}(g).$$

Since $g^{-*}e = \Theta(g)e = (ge)^{-1}$ [FK94, Theorem III.5.3], Φ can be written, as a function on Ω ,

$$\Phi(x) = \int_K \Delta_{-(i\nu+\rho'_l)}(x^{-1}) dk \Delta^{-\frac{ld}{4}}(x) = \Phi_{-(i\nu+\rho'_l)}(x^{-1}) \Delta^{-\frac{ld}{4}}(x),$$

and since $\Phi_{-(i\nu+\rho'_l)}(x^{-1}) = \Phi_{i\nu+\rho'_l+2\rho}(x)$ with $\rho = \sum_{j=1}^r (2j-r-1)\delta_j$ [FK94, Theorem XIV.3.1 (iv)],

$$\Phi(x) = \Phi_{i\nu+\eta_l+\rho}(x) \quad \text{where} \quad \eta_l = \frac{d}{4} \sum_{j=l+1}^r (2j-l-r-1)\delta_j.$$

Recall [FK94, Theorem XIV.3.1 (iii)] that $\Phi_{\nu'+\rho} = \Phi_{\nu+\rho}$ if and only if $\nu' = w\nu$ for $w \in W$. Hence the representations π_{λ} and $\pi_{\lambda'}$, with λ, λ' in \mathfrak{a}_l^* , are equivalent if and only if $i\lambda' + \eta_l = w(i\lambda + \eta_l)$. Since η_l is real it follows that $\pi_{\lambda'}$ is equivalent to π_{λ} if and only if $\lambda' = w\lambda$ with $w \in W_l := \mathfrak{S}_l$.

5. THE INTERTWINING OPERATOR AND THE PLANCHEREL FORMULA

Let $\nu \in \mathbb{C}^l$ such that $\Re(-(i\nu + \rho'_l)) \geq 0$. Then for $f \in C_0^\infty(\partial_l \Omega)$ and $g \in G$, the formula

$$(10) \quad T_\nu f(g) = \int_{\partial_l \Omega} f(x) \Delta_{-(i\nu+\rho'_l)}(g^*x) d\mu_l(x)$$

defines a continuous function on G . Moreover, it follows from (2) and Lemma 4.4 that

$$T_\nu f \in C(G, Q_l, 1 \otimes e^{i\nu+\rho'_l} \otimes 1).$$

We will also view T_ν as an operator with values in $C(G, Q_l, 1 \otimes e^{i\nu+\rho'_l} \otimes 1) \otimes \mathbb{C}$ (in the obvious way), and hence in \mathcal{H}_ν .

Lemma 5.1. *For $\Re(-(i\nu + \rho'_l)) \geq 0$, the operator*

$$T_\nu : C_0^\infty(\partial_l \Omega) \rightarrow \mathcal{H}_\nu$$

intertwines π^l and π_ν .

Proof. Let us set $\nu' = -(i\nu + \rho'_l)$. Let $h \in G$. Then

$$\begin{aligned} T_\nu(h.f)(g) &= \int_{\partial \Omega_l} \Delta^{\frac{ld}{4}}(h) f(h^*x) \Delta_{\nu'}(g^*x) d\mu_l(x) \\ &= \Delta^{-\frac{ld}{4}}(h) \int_{\partial \Omega_l} f(x) \Delta_{\nu'}(g^*h^{-*}x) d\mu_l(x) \\ &= \Delta^{-\frac{ld}{4}}(h) \int_{\partial \Omega_l} f(x) \Delta_{\nu'}((h^{-1}g)^*x) d\mu_l(x), \end{aligned}$$

i.e.,

$$(11) \quad T_\nu(h.f)(g) = \Delta^{-\frac{ld}{4}}(h) T_\nu(f)(h^{-1}g).$$

□

Since $\rho'_l = \frac{d}{2} \sum_{j=1}^l (r+l+1-2j)\delta_j$, we do not have $\Re((-i\lambda + \rho'_l)) \geq 0$ when $\lambda \in \mathfrak{a}_l^*$, and the integral (10) does not converge. This means that the integral has to be interpreted in a suitable sense using analytic continuation in the parameter ν . For this we recall that when $\nu \in \mathbb{C}^l$, the Riesz distribution

$R_{\nu + \frac{ld}{2}}$ on V can be defined as the analytic continuation of the following integral

$$R_{\nu + \frac{ld}{2}}(F) = \Gamma_{\Omega^{(l)}} \left(\nu + \frac{ld}{2} \right)^{-1} \int_{\partial_l \Omega} F(x) \Delta_{\nu}(x) d\mu_l(x), \quad F \in \mathcal{S}(V),$$

where $\mathcal{S}(V)$ is the Schwartz space of V , and that it has support in $\overline{\partial_l \Omega}$ (cf. [Ish00, Theorem 5.1 and 5.2], where the integral is actually defined over $\mathcal{O}_l = \{x \in \partial_l \Omega \mid \Delta_{(l)}(x) \neq 0\}$, but $\mu_l(\partial_l \Omega \setminus \mathcal{O}_l) = 0$). The restriction (denoted by the same symbol) to the open set $V_{\geq l}$ is then a distribution with support in the submanifold $\partial_l \Omega$. We can therefore consider the vertical restrictions $R_{\nu + \frac{ld}{2}}|_{\partial_l \Omega}$ (cf. Appendix A). Since $R_{\nu + \frac{ld}{2}}$ is a measure with support on $\partial_l \Omega$ for $\Re \nu \geq 0$, these restrictions do not depend on the choice of a tubular neighbourhood (cf. Proposition A.9). For $\Re(-(i\nu + \rho'_l)) \geq 0$, we have

$$T_{\nu} f(g) = \Gamma_{\Omega^{(l)}} \left(-(i\nu + \rho'_l) + \frac{ld}{2} \right) \Delta^{-\frac{ld}{2}}(g) \left(R_{-(i\nu + \rho'_l) + \frac{ld}{2}} \right) |_{\partial_l \Omega} (f(g^{-*} \cdot)).$$

Hence, we can let the right hand side define an analytic continuation of the integrals $T_{\nu} f(g)$. It is defined on the complement \mathcal{Z} of the set of poles of the meromorphic function $\Gamma_{\Omega^{(l)}} \left(-(i\nu + \rho'_l) \right)$. Since for fixed $f \in C_0^{\infty}(\partial_l \Omega)$ the map $g \mapsto f(g^{-*} \cdot)$ is continuous, the function $g \mapsto T_{\nu}(f)(g)$ is continuous.

Proposition 5.2. *For any $\nu \in \mathcal{Z}$,*

$$T_{\nu} f \in C(G, Q_l, 1 \otimes e^{i\nu + \rho'_l} \otimes 1),$$

and the operator

$$T_{\nu} : C_0^{\infty}(\partial_l \Omega) \rightarrow \mathcal{H}_{\nu}$$

intertwines π^l and π_{ν} .

Proof. The equation describing the Q_l -equivariance as well as eq. (11) are analytic in the parameter ν . Hence they hold by analytic continuation since they hold on the open set where $\Re(-(i\nu + \rho'_l)) > 0$. \square

We now recall, in order to fix the notations, the definition of the spherical Fourier transform on $\Omega^{(l)}$. If f is a continuous function with compact support on $\Omega^{(l)}$ which is $K^{(l)}$ -invariant, its spherical Fourier transform is

$$\widehat{f}(\nu) = \int_{\Omega^{(l)}} f(x) \Phi_{-\nu + \rho^{(l)}}^{(l)}(x) d_*^{(l)} x,$$

where $\nu \in (\mathfrak{a}_{\mathbb{C}})^*$ and $\rho^{(l)} = \frac{d}{4} \sum_{j=1}^l (2j - l - 1) \delta_j$. Since f has compact support, the function \widehat{f} is holomorphic on $(\mathfrak{a}_{\mathbb{C}})^*$. For latter use we also recall the inversion formula for $f \in C_0^{\infty}(\Omega^{(l)})^{K^{(l)}}$ and $\lambda \in \mathfrak{a}_l^*$ (cf. [FK94, Theorem XIV.5.3] and [Hel84, Ch. III, Theorem 7.4]):

$$(12) \quad f(x) = c_0^{(l)} \int_{\mathfrak{a}_l^*} \widehat{f}(i\lambda) \Phi_{i\lambda + \rho^{(l)}}^{(l)}(x) \frac{d\lambda}{|c^{(l)}(\lambda)|^2},$$

where $d\lambda$ is the Lebesgue measure on $\mathfrak{a}_l^* \simeq \mathbb{R}^l$, $c^{(l)}(\lambda)$ is Harish Chandra's c -function for $\Omega^{(l)}$, and $c_0^{(l)}$ is a positive constant.

Now let $f \in C_0^{\infty}(\partial_l \Omega)^K$, and observe that $\Omega^{(l)}$, being a fibre of $\partial_l \Omega \rightarrow \Pi_l$, is closed in $\partial_l \Omega$, and hence $f|_{\Omega^{(l)}}$ (sometimes still denoted by f) has compact

support in $\Omega^{(l)}$. Moreover, since any $k \in K^{(l)}$ extends to an element of K , the function $f|_{\Omega^{(l)}}$ is $K^{(l)}$ -invariant.

Proposition 5.3. *If $f \in C_0^\infty(\partial_l \Omega)$ is K -invariant then*

$$(13) \quad T_{\nu} f(g) = \gamma_{-(i\nu + \rho_l)}^{(l)} \widehat{f}(i\nu - \frac{rd}{4}) \Delta_{-(i\nu + \rho_l)}(g^* e)$$

Proof. Let us set $\nu' = -(i\nu + \rho_l)$ in the following. Again, by analytic continuation, it suffices to prove the equality for $\Re(\nu') \geq 0$. Since f and μ_l are K -invariant one has for all k in K ,

$$T_{\nu} f(g) = \int_{\partial \Omega_l} f(k^{-1}x) \Delta_{\nu'}(g^* x) d\mu_l(x) = \int_{\partial \Omega_l} f(x) \Delta_{\nu'}(g^* kx) d\mu_l(x).$$

Hence

$$T_{\nu} f(g) = \int_K T_{\nu} f(g) dk = \int_{\partial \Omega_l} f(x) \left(\int_K \Delta_{\nu'}(g^* kx) dk \right) d\mu_l(x).$$

Writing $g^* = th$, $t \in NA$, $h \in K$, we have

$$\Delta_{\nu'}(thkx) = \Delta_{\nu'}(h kx) \Delta_{\nu'}(te) = \Delta_{\nu'}(h kx) \Delta_{\nu'}(g^* e),$$

and using the left invariance of the Haar measure of K ,

$$\begin{aligned} \int_K \Delta_{\nu'}(g^* kx) dk &= \int_K \Delta_{\nu'}(h kx) dk \Delta_{\nu'}(g^* e) \\ &= \int_K \Delta_{\nu'}(kx) dk \Delta_{\nu'}(g^* e) \\ &= \Phi_{\nu'}(x) \Delta_{\nu'}(g^* e), \end{aligned}$$

hence,

$$T_{\nu} f(g) = \int_{\Omega^{(l)}} f(x) \Phi_{\nu'}(x) d\mu_l(x) \Delta_{\nu'}(g^* e).$$

Now Upmeyer and Arazy's polar decomposition (4) for μ_l yields

$$T_{\nu} f(g) = \int_{\Omega^{(l)}} f(x) \Phi_{\nu'}(x) \Delta_{\frac{rd}{2}}^{(l)}(x) d_*^{(l)} x \Delta_{\nu'}(g^* e),$$

and by Theorem 3.1,

$$\begin{aligned} T_{\nu} f(g) &= \gamma_{\nu'}^{(l)} \int_{\Omega^{(l)}} f(x) \Phi_{\nu'}^{(l)}(x) \Delta_{\frac{rd}{2}}^{(l)}(x) d_*^{(l)} x \Delta_{\nu'}(g^* e) \\ &= \gamma_{\nu'}^{(l)} \int_{\Omega^{(l)}} f(x) \Phi_{\nu' + \frac{rd}{2}}^{(l)}(x) d_*^{(l)} x \Delta_{\nu'}(g^* e) \\ &= \gamma_{\nu'}^{(l)} \widehat{f}(-\nu' + \rho^{(l)} - \frac{rd}{2}) \Delta_{\nu'}(g^* e). \end{aligned}$$

Since $-\rho_l' + \rho^{(l)} - \frac{rd}{2} = -\frac{rd}{4}$, we eventually get

$$T_{\nu} f(g) = \gamma_{-(i\nu + \rho_l)}^{(l)} \widehat{f}(i\nu - \frac{rd}{4}) \Delta_{-(i\nu + \rho_l)}(g^* e).$$

□

In the following we set,

$$\widetilde{f}(\nu) = \gamma_{-(i\nu + \rho_l)}^{(l)} \widehat{f}(i\nu - \frac{rd}{4}),$$

and we note that it defines a meromorphic function whose poles are those of $\Gamma_\Omega(-i\nu + \rho'_l + \frac{ld}{2})$. The following lemma will be used in the proof of the Plancherel formula.

Lemma 5.4 (Inversion formula). *Let $f \in C_0^\infty(\partial_l \Omega)^K$ and let $\lambda \in \mathfrak{a}_l^*$. Then for $x \in \Omega^{(l)}$,*

$$f(x) = c_0^{(l)} \int_{\mathfrak{a}_l^*} \tilde{f}(\lambda) \Phi_{i\lambda + \rho^{(l)} - \frac{rd}{4}}^{(l)}(x) (\gamma_{-(i\lambda + \rho'_l)}^{(l)})^{-1} \frac{d\lambda}{|c^{(l)}(\lambda)|^2}.$$

Proof. We have

$$(14) \quad \widehat{f \Delta_{(l)}^{\frac{rd}{4}}}(i\lambda) = \widehat{f}(i\lambda - \frac{rd}{4}) = \tilde{f}(\lambda) (\gamma_{-(i\lambda + \rho'_l)}^{(l)})^{-1},$$

hence the inversion formula (12) applied to the function $f \Delta_{(l)}^{\frac{rd}{4}}$ gives the desired formula. \square

We now state the main result of the article. Recall the notations from the end of section 4.

Theorem 5.5 (The Plancherel Theorem). *Let p be the measure on \mathfrak{a}_l^*/W_l defined by*

$$dp(\lambda) = \frac{c_0^{(l)}}{|\gamma_{-(i\lambda + \rho'_l)}^{(l)} c^{(l)}(\lambda)|^2} d\lambda.$$

Then there exists an isomorphism of unitary representations

$$T : (\pi^l, L^2(\partial_l \Omega)) \simeq \left(\int_{\mathfrak{a}_l^*} \pi_\lambda dp(\lambda), \int_{\mathfrak{a}_l^*} \mathcal{H}_\lambda dp(\lambda) \right),$$

such that for every $f \in C_0^\infty(\partial_l \Omega)$, $(Tf)_\lambda = T_\lambda f$.

Proof. First we prove that for any K -invariant function f in $C_0^\infty(\partial_l \Omega)$,

$$(15) \quad \int_{\partial_l \Omega} |f(x)|^2 d\mu_l(x) = \int_{\mathfrak{a}_l^*} |\tilde{f}(\lambda)|^2 dp(\lambda).$$

For this purpose we use the polar decomposition for μ_l and the inversion formula of Lemma 5.4. Then

$$\begin{aligned} \int_{\partial_l \Omega} |f(x)|^2 d\mu_l(x) &= \int_{\Omega^{(l)}} |f(x)|^2 \Delta_{(l)}^{\frac{rd}{2}}(x) d_*^{(l)} x \\ &= \int_{\Omega^{(l)}} f(x) \Delta_{(l)}^{\frac{rd}{2}}(x) c_0^{(l)} \int_{\mathfrak{a}_l^*} \overline{\tilde{f}(\lambda)} \Phi_{i\lambda + \rho^{(l)} - \frac{rd}{4}}^{(l)}(x) (\gamma_{-(i\lambda + \rho'_l)}^{(l)})^{-1} \frac{d\lambda}{|c^{(l)}(\lambda)|^2} d_*^{(l)} x \\ &= \int_{\mathfrak{a}_l^*} \overline{\tilde{f}(\lambda) (\gamma_{-(i\lambda + \rho'_l)}^{(l)})^{-1}} \left(\int_{\Omega^{(l)}} f(x) \Delta_{(l)}^{\frac{rd}{4}}(x) \Phi_{-i\lambda + \rho^{(l)}}^{(l)}(x) d_*^{(l)} x \right) \frac{c_0^{(l)} d\lambda}{|c^{(l)}(\lambda)|^2} \\ &= \int_{\mathfrak{a}_l^*} \overline{\tilde{f}(\lambda) (\gamma_{-(i\lambda + \rho'_l)}^{(l)})^{-1}} \widehat{f \Delta_{(l)}^{\frac{rd}{4}}}(i\lambda) \frac{c_0^{(l)} d\lambda}{|c^{(l)}(\lambda)|^2} \\ &= \int_{\mathfrak{a}_l^*} |\tilde{f}(\lambda)|^2 \frac{c_0^{(l)} d\lambda}{|\gamma_{-(i\lambda + \rho'_l)}^{(l)} c^{(l)}(\lambda)|^2}. \end{aligned}$$

In the last equality we have used again the formula (14).

The next step is to prove that for a dense subset of functions f in $C_0^\infty(\partial_l\Omega)$, the identity

$$\int_{\partial_l\Omega} |f(x)|^2 d\mu_l(x) = \int_{\mathfrak{a}_l^*} |T_\lambda f|_\lambda^2 dp(\lambda)$$

holds.

Recall that $L^1(G)$ is a Banach $*$ -algebra when equipped with convolution as multiplication, and $\varphi^*(g) := \overline{\varphi(g^{-1})}$. Let $L^1(G)^\#$ denote the (commutative) closed subalgebra of left and right K -invariant functions in $L^1(G)^\#$. There is a natural projection $L^1(G) \rightarrow L^1(G)^\#$,

$$\varphi \mapsto \varphi^\# := \int_K \int_K \varphi(k_1^{-1} \cdot k_2) dk_1 dk_2.$$

For a unitary representation (τ, \mathcal{H}) of G , there is a $*$ -representation (also denoted by τ) of $L^1(G)$ on \mathcal{H} given by

$$\tau(\varphi)v := \int_G \varphi(g)\tau(g)v dg, \quad v \in \mathcal{H}.$$

The representations of K and $L^1(G)$ are related by

$$(16) \quad \tau(k_1)\tau(\varphi)\tau(k_2) = \tau(\varphi(k_1^{-1} \cdot k_2^{-1})), \quad \varphi \in L^1(G), \quad k_1, k_2 \in K.$$

The subspace \mathcal{H}^K of K -invariants is invariant under $L^1(G)^\#$. From (16), it follows that for any $\varphi \in L^1(G)$, and $u, v \in \mathcal{H}^K$,

$$(17) \quad \langle \tau(\varphi)u, v \rangle = \langle \tau(\varphi^\#)u, v \rangle.$$

Let ξ be the K -invariant cyclic vector in $L^2(\partial_l\Omega)$. We claim that there exists a sequence $\{\xi_n\}_{n=1}^\infty \subseteq C_0^\infty(\partial_l\Omega)^K$, such that $\xi_n \rightarrow \xi$ in $L^2(\partial_l\Omega)$. To see this, we can first choose a sequence $\{\zeta_n\}_{n=1}^\infty \subseteq C_0^\infty(\partial_l\Omega)$ that converges to ξ . Next, observe that the orthogonal projection $P : L^2(\partial_l\Omega) \rightarrow L^2(\partial_l\Omega)$ is given by $f \mapsto \int_K f(k^{-1} \cdot) dk$. Then $P(f)$ is smooth if f is smooth. Moreover, $\text{supp } f$ is contained in the image of the map $K \times \text{supp } f \rightarrow \partial_l\Omega$, $(k, x) \rightarrow kx$. It follows that $P(C_0^\infty(\partial_l\Omega)) \subseteq C_0^\infty(\partial_l\Omega)^K$. Hence, the claim holds with $\xi_n := P(\zeta_n)$. The subspace

$$\mathcal{H}_0 := \{\pi^l(f)\xi_n \mid f \in C_0^\infty(G), n \in \mathbb{N}\}$$

is then dense in $L^2(\partial_l \Omega)$. For $\varphi \in C_0^\infty(G)$, $n \in \mathbb{N}$, we have, by (17) and (15),

$$\begin{aligned}
\langle \pi^l(\varphi)\xi_n, \pi^l(\varphi)\xi_n \rangle_{L^2(\partial_l \Omega)} &= \langle \pi^l(\varphi^* * \varphi)\xi_n, \xi_n \rangle_{L^2(\partial_l \Omega)} \\
&= \langle \pi^l((\varphi^* * \varphi)^\#)\xi_n, \xi_n \rangle_{L^2(\partial_l \Omega)} \\
&= \int_{\mathfrak{a}_l^*} \langle T_\lambda(\pi^l((\varphi^* * \varphi)^\#)\xi_n), T_\lambda(\xi_n) \rangle_\lambda dp(\lambda) \\
&= \int_{\mathfrak{a}_l^*} \langle \pi_\lambda((\varphi^* * \varphi)^\#)T_\lambda(\xi_n), T_\lambda(\xi_n) \rangle_\lambda dp(\lambda) \\
&= \int_{\mathfrak{a}_l^*} \langle \pi_\lambda(\varphi^* * \varphi)T_\lambda(\xi_n), T_\lambda(\xi_n) \rangle_\lambda dp(\lambda) \\
&= \int_{\mathfrak{a}_l^*} \langle \pi_\lambda(\varphi)T_\lambda(\xi_n), \pi(\varphi)T_\lambda(\xi_n) \rangle_\lambda dp(\lambda) \\
&= \int_{\mathfrak{a}_l^*} \langle T_\lambda(\pi^l(\varphi)\xi_n), T_\lambda(\pi^l(\varphi)\xi_n) \rangle_\lambda dp(\lambda).
\end{aligned}$$

Hence, the operator T defined on \mathcal{H}_0 by $T(\pi^l(\varphi)\xi_n) = (\pi_\lambda(\varphi)T_\lambda(\xi_n))_\lambda$ extends uniquely to a G -equivariant isometric operator

$$T : L^2(\partial_l \Omega) \rightarrow \int_{\mathfrak{a}_l^*} \mathcal{H}_\lambda dp(\lambda).$$

It now only remains to prove the surjectivity of T . Assume therefore that $(\eta_\lambda)_\lambda$ is orthogonal to the image of T . Then for all φ in $L^1(G)$ and $h \in L^1(G)^\#$,

$$\int_{\mathfrak{a}_l^*} \langle \pi_\lambda(\varphi * h)(T\xi)_\lambda, \eta_\lambda \rangle_\lambda dp(\lambda) = 0,$$

i.e.,

$$\int_{\mathfrak{a}_l^*} \check{h}(\lambda) \langle \pi_\lambda(\varphi)(T\xi)_\lambda, \eta_\lambda \rangle_\lambda dp(\lambda) = 0,$$

where $\check{h}(\lambda)$ is the Gelfand transform of h restricted to \mathfrak{a}_l^* . Recall that the set of bounded spherical functions can be identified with the character space of $L^1(G)^\#$, and hence the image of $L^1(G)^\#$ under the Gelfand transform separates points in this space. It thus follows from the Stone-Weierstrass Theorem that the functions \check{h} are dense in the space of continuous functions on \mathfrak{a}_l^* that are invariant under the action of W_l . Hence $\langle \pi_\lambda(f)(T\xi)_\lambda, \eta_\lambda \rangle_\lambda = 0$ p -almost everywhere. By separability of $L^1(G)$, there is a set U with $p(\mathfrak{a}_l^* \setminus U) = 0$ such that for all f in $L^1(G)$ and $\lambda \in U$, $\langle \pi_\lambda(f)(T\xi)_\lambda, \eta_\lambda \rangle_\lambda = 0$. By cyclicity of $(T\xi)_\lambda$ (note that $(T\xi)_\lambda$ is non-zero p -almost everywhere), η_λ is zero p -almost everywhere. \square

Remark 5.6. We want to point out that it is actually not necessary to prove the analytic continuation of T_ν (and hence to use the theory of Riesz distributions) to derive the decomposition of π^l (however, the natural operator T above is then replaced by an abstract one). Indeed, by the Cartan-Helgason theorem ([Hel84, Ch. III, Lemma 3.6]) we have $\mathcal{H}_\lambda^K = \mathbb{C} v_\lambda$ when $\lambda \in \mathfrak{a}_l^*$, and hence we can set

$$T_\lambda : C_0^\infty(\partial_l \Omega)^K \rightarrow \mathcal{H}_\lambda^K, \quad f \mapsto \tilde{f}(\lambda) v_\lambda,$$

and by (15) we thus obtain an operator $T : L^2(\partial_l \Omega)^K \rightarrow \int_{\mathfrak{a}_l^*} \mathcal{H}_{\lambda}^K dp(\lambda)$. Assume that we can prove that T intertwines the actions of $C_0^\infty(G)^\#$. Then for $\varphi \in C_0^\infty(G)^\#$,

$$\begin{aligned} \langle \pi^l(\varphi)\xi, \xi \rangle &= \langle T\pi^l(\varphi)\xi, T\xi \rangle = \langle \pi(\varphi)T\xi, T\xi \rangle \\ &= \int_{\mathfrak{a}_l^*} \langle \pi_{\lambda}(\varphi)(T\xi)_{\lambda}, (T\xi)_{\lambda} \rangle_{\lambda} dp(\lambda) \\ &= \int_{\mathfrak{a}_l^*} \check{\varphi}(\lambda) |(T\xi)_{\lambda}|_{\lambda}^2 dp(\lambda), \end{aligned}$$

where $\check{\varphi}$ is defined by $\pi_{\lambda}(\varphi)v_{\lambda} = \check{\varphi}(\lambda)v_{\lambda}$. The proof of [Sep07b, Theorem 10] shows that the decomposition of π^l then follows. We now prove the intertwining property. It is equivalent to the equality

$$(18) \quad \widetilde{\pi^l(\varphi)f(\lambda)} = \tilde{f}(\lambda)\check{\varphi}(\lambda).$$

Let $\nu \in \mathbb{C}^l$. Then for $f \in C_0^\infty(\partial_l \Omega)$ and $\varphi \in C_0^\infty(G)^\#$ we have, where $\nu' = -(i\nu + \rho_l')$,

$$\begin{aligned} \pi_{\nu}(\varphi) (\Delta_{\nu'}((\cdot)^*e) \otimes 1)(g) &= \int_G \varphi(h) \Delta_{\nu'}(g^*h^{-*}e) \otimes \Delta^{-\frac{ld}{4}}(he) dh \\ &= \int_G \varphi(h) \Delta^{-\frac{ld}{4}}(he) \left(\int_K \Delta_{\nu'}(g^*kh^{-*}e) dk \right) \otimes 1 dh \\ &= \int_G \varphi(h) \Delta^{-\frac{ld}{4}}(he) \left(\int_K \Delta_{\nu'}(kh^{-*}e) dk \right) dh (\Delta_{\nu'}(g^*e) \otimes 1) \\ &= \int_G \varphi(h) \Delta^{-\frac{ld}{4}}(he) \Phi_{\nu'}(h^{-*}e) dh (\Delta_{\nu'}(g^*e) \otimes 1), \end{aligned}$$

i.e.,

$$\pi_{\nu}(\varphi)v_{\nu} = \check{\varphi}(\nu)v_{\nu},$$

where $\check{\varphi}(\nu)$ is holomorphic on \mathbb{C}^l . If $\Re(-(i\nu + \rho_l')) \geq 0$, then, by Lemma 5.1, the operator T_{ν} intertwines the actions of $C_0^\infty(G)^\#$, and hence

$$\widetilde{\pi^l(\varphi)f(\nu)} = \tilde{f}(\nu)\check{\varphi}(\nu).$$

Thus (18) follows by analytic continuation.

APPENDIX A. RESTRICTIONS OF DISTRIBUTIONS

Let X be a smooth n -dimensional manifold. Let $\mathcal{D}(X)$ denote the space of compactly supported smooth functions on X , i.e., the test functions on X . For any chart (V, ϕ) , compact subset $K \subseteq \tilde{V} := \phi(V)$, and $N \in \mathbb{N}$, consider the seminorm

$$(19) \quad p_{V,K,N}(f) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq N} \sup_{x \in K} |D^{\alpha}(f \circ \phi^{-1})(x)|$$

on the space of smooth functions on X . Here $D^{\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$, and $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$. For a compact set $K \subseteq X$, let $\mathcal{D}(K)$ be the space of smooth functions with support in K equipped with the topology induced by the above seminorms. We recall that a distribution on X is a continuous functional on $\mathcal{D}(X) = \cup_K \mathcal{D}(K)$ equipped with the

inductive limit topology. We let $\mathcal{D}'(X)$ denote the space of distributions on X . Let $\{U_i\}$ be an open cover of X . Then a linear functional on $\mathcal{D}(X)$ is continuous if and only if its restriction to every $\mathcal{D}(U_i)$ is continuous.

We will now construct restrictions to a closed submanifold Y of distributions that have support on Y . To have a well-defined notion of restriction, one can not permit arbitrary extensions to X of test functions on Y . Instead, we will require the extension to be locally constant along some predescribed direction. This can be made precise using tubular neighbourhoods.

Definition A.1. Let X be a smooth n -dimensional manifold, and let Y be a k -dimensional submanifold. A *tubular neighbourhood* of Y in X consists of a smooth vector bundle $\pi : E \rightarrow Y$, an open neighbourhood Z of the image, $\zeta_E(Y)$, of the zero section in E , and a diffeomorphism $f : Z \rightarrow O$ onto an open set $O \subseteq X$ containing Y , such that the diagram

$$\begin{array}{ccc} & Z & \\ \zeta_E \uparrow & \searrow f & \\ Y & \xrightarrow{j} & X \end{array}$$

commutes. Here $j : Y \rightarrow X$ is the inclusion map.

Remark A.2. Any closed submanifold Y of X admits a tubular neighbourhood (cf. [Lan72, Ch. IV, F, Thm. 9]). Any splitting of the tangent bundle of X over Y , $T(X)|_Y = T(Y) \oplus E$, gives such a vector bundle E . In particular, given a Riemannian metric on X , E can be chosen as the orthogonal complement to $T(Y)$ in $T(X)|_Y$.

Since the concepts we are dealing with are of a local nature we can without loss of generality assume that $X = Z$ itself is a tubular neighbourhood of Y .

Definition A.3. A function f on X is said to be *locally vertically constant* around Y , l.v.c., if for any $x \in Y$, there exists an open neighbourhood W_x of x in X such that for $y \in W_x$, $f(y) = f(\pi(y))$. Moreover, if g is a function on Y , and $f|_Y = g$, f is called an l.v.c. extension of g .

Lemma A.4.

- (i) Any test function φ on Y admits a l.v.c extension $\tilde{\varphi} \in \mathcal{D}(X)$.
- (ii) An l.v.c. function f that vanishes on Y vanishes on some neighbourhood of Y .

Proof. Since $\text{supp } \varphi$ is compact, it can be covered by finitely many open neighbourhoods O_1, \dots, O_N diffeomorphic to products $U_i \times V_i \subset \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, where U_i is open in \mathbb{R}^k and V_i is an open neighbourhood of 0 in \mathbb{R}^{n-k} , in such a way that π corresponds to the projection onto the first coordinate. Let (ψ_i) be a smooth partition of unity on $U = \bigcup_{i=1}^N \pi(O_i)$ subordinate to the cover $\pi(O_i)$, $i = 1, \dots, N$, and let $\varphi_i = \varphi \psi_i$. Then each φ_i can be identified with a test function on U_i , and by multiplying this with a test function on V_i which is 1 on some neighbourhood of 0, we obtain a l.v.c. extension $\tilde{\varphi}_i$ of φ_i . Then $\tilde{\varphi} = \sum \tilde{\varphi}_i$ is a l.v.c extension of φ . This proves (i). For (ii) just observe that if f is an l.v.c function that vanishes on Y , then Y is in the complement of the support of f . \square

Assume now that $u \in \mathcal{D}'(X)$ has support on the submanifold Y . Then l.v.c. test functions that vanish on a neighbourhood of Y are in the kernel of u , and the preceding lemma enables us to make the following definition.

Definition A.5. Let $u \in \mathcal{D}'(X)$ be a distribution with support on Y . The vertical restriction, $u|_Y$, of u to Y is the unique distribution on Y that satisfies

$$u|_Y(\varphi|_Y) = u(\varphi),$$

for any $\varphi \in \mathcal{D}(X)$ which is l.v.c. around Y .

To see that the functional $u|_Y$ really is a distribution on Y , it suffices to verify the continuity for test functions with support in trivialising open sets. In this case the verification is straightforward using the l.v.c extension from the proof of Lemma A.4.

Remark A.6. Note that the vertical restriction depends on the choice of tubular neighbourhood, i.e., on a choice of complement E in the splitting of vector bundles in Remark A.2. However, when u is a measure on X with support on Y , the vertical restriction $u|_Y$ is u itself, now viewed as a distribution on Y .

We now consider holomorphic families of distributions and their properties under restriction.

Definition A.7. Let $\Omega \subseteq \mathbb{C}^m$ be an open set, and let $\{u^z\}_{z \in \Omega}$ be a family of distributions on the smooth manifold X . Then this family is called a *holomorphic family of distributions* if the map $z \mapsto u^z(\varphi)$ is holomorphic on Ω for every $\varphi \in \mathcal{D}(X)$.

Remark A.8. It follows immediately from Definition A.5 that if $\{u^z\}_{z \in \Omega}$ is a holomorphic family of distributions with support on Y , then the family $\{u^z|_Y\}_{z \in \Omega}$ is a holomorphic family of distributions on Y .

Proposition A.9. Let $\Omega \subseteq \mathbb{C}^n$ be open and connected, and let $\{u^z\}_{z \in \Omega}$ be a holomorphic family of distributions on X with support on Y . Assume that there exists an open subset $U \subseteq \Omega$, such that u^z is a measure with support on Y for $z \in U$. Then the whole family $\{u^z|_Y\}_{z \in \Omega}$ is independent of E .

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